

A Multivariable Grey Prediction Model with Fractional-Order Time-Delay Parameters and Its Application

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Abstract: Integer-order accumulating generation used by the classical multivariable grey prediction model is unable to adapt to fluctuating sequences and the contemporaneous modeling assumption fails to consider causal time delays between variables. In this paper we construct a fractional-order time-delay multivariable grey prediction model that incorporates fractional-order accumulating generation and multivariable time-delay parameters into one model. Based on it, we propose a collaborative identification strategy for order and time delay, an improved least squares estimation as well as a recursive dynamic parameter adjustment scheme. We have evaluated performance of our model from the three aspects of fitting accuracy, prediction robustness and decomposition of improvement contributions, and defined its applicable conditions. Our model enhances the capability of grey prediction theory for dealing with multivariable sequences with time-delay features.

Keywords: fractional-order accumulating generation; time-delay parameters; multivariable grey prediction model; parameter optimization; applicable boundary

Introduction

The Grey prediction models have great advantages for predicting small sample and information incomplete systems. The multivariable grey prediction models using information from several driving variables together to predict behavior variables which are widely used. The classical models were set up by first-order accumulating generation assuming that there exists synchronous action. First order accumulation is not enough to make the sequence smoother if it has high fluctuations; synchronous assumption neglects time delay in information transmission between variables resulting deviation from the true dynamic structure in systems with long transmission path. There are still some gaps regarding improvement researches on joint optimization of accumulation methods and lag structures. We introduced fractional-order accumulating generation and time-delay parameters constructing a prediction model adapted to complex data characteristic characterizing the delay relationship among variables solving problems about parameter identification, performance evaluation and defining the applicable boundary complementing the methodology system of grey prediction theory.

1. Construction Mechanism of the Fractional-Order Time-Delay Multivariable Grey Prediction Model

1.1 Regulatory Mechanism of Fractional-Order Accumulating Generation Operation on the Smoothness of the Original Sequence

Fractional-order accumulating generation operator is an extension of the traditional integer-order accumulating generation operator naturally; it's most valuable feature is that it extended the accumulation order from discrete integers into a continuously variable real number. The traditional first-order accumulating generation equally weight every piece of history data. It works perfectly for sequences which have good smoothness but when facing multivariable sequences with strong fluctuation, equal-weight accumulation generally doesn't work efficiently to suppress random disturbance nor the exponential law characteristics of the generated sequence are not significant enough. With the flexibility of adjusting order, we can tune the accumulation intensity flexibly between 0-1st order (even higher than first order) which matches the inherent smoothness characteristics of different sequences^[1].

If the order value is relatively small, the smoothing effect caused by the accumulation operation on the original fluctuation cannot be significant so that the short term change information in the sequence still exists. With the increase of the order value (the increasing cumulative weight of the accumulation operation on the historical information), the overall trend of the generated sequence will become more obvious. Since the feature of continuously adjustable order allows the fractional-order accumulating generation to adjust its accumulation speed based on the structure of the sequence itself, ensuring that it can achieve a balance between keeping the effective information of the original sequence and improving the smoothness of the generated sequence, guaranteeing the data foundation level for later modeling.

1.2 Characterization of the Time-Delay Structure of Multivariable Driving Information under the Introduction of Lag Effects

Basic multivariable grey prediction model presupposes that the correlation between the system's behavioral variable with any one driving variable has a contemporaneous correspondence relationship (i.e., the driving variable immediately full impact on the behavioral variable if it exists at the present moment). We ignore the time consumed for the information transmission and effect exertion among variables. If there are obvious transmission delays from a driving variable to the behavioral variable, then our model's description about the dynamic structure of the system will be systematically misaligned by its temporal order. By introducing time-delay parameters we can correct such misalignment; allowing the influence of every driving variable on the behavioral variable happens after some periods so as to more truthfully reflect the causal time-delay relationships among variables. Because of their own properties and transmission path difference, different driving variables usually have different time-delay length in influencing the same behavioral variable. Therefore, independent time-delay parameters need to be set for each driving variable in the model.

Sometimes there are cumulative influences from a driving variable at several consecutive lags; it needs that the time-delay structure has the capability to describe the persistence characteristics of influence. Reasonable settings of time-delay structure will decide whether the model could correctly identify the dynamic response relationship between driving variables and the behavioral variable. It is an inevitable key link for multivariable grey modeling.

1.3 Structural Design of the Grey Model Coupled with Fractional-Order Time-Delay Parameters

Here lies the question about integrating both kinds of mechanisms (fractional-order accumulating generation and time-delay parameters) into our model. The target of the accumulating generation mechanism is the time series data of one variable; aiming to improve the smoothness characteristic of the sequence. The target of the time delay mechanism is the causal correspondence relationship among variables; aiming to correct the deviation in the timing of action among variables. Both kinds of mechanisms operate at different levels during the modeling process. So we need to clarify their execution order when designing the coupling. A reasonable way would be executing fractional-order accumulating generation on each variable sequence separately first then construct a driving structure with lag relationships at the level of the accumulated sequences so as to ensure that the selection of time-delay parameters will not be affected by the random fluctuations in the original sequences but also ensures that the improving effect of the accumulating generation on smoothness could be fully transferred to the overall prediction framework of the model^[2].

After designing the coupling, the form of our model equation is similar as the classical multivariable grey model; however, we have greatly extended the connotation of the system development coefficient and the driving coefficients and enhanced its capability in adapting to the data characteristics. Identification for the model parameters requires solution of a mixed estimation problem involving both linear and nonlinear parameters existing together. In this set of parameters, the fractional order and the time-delay length of each driving variable are nonlinear parameters whereas the system development coefficient and the driving coefficients are linear parameters. Each type has different optimization goal and constraint which needs to be solved collaboratively according to an overall optimization criterion.

2. Parameter Optimization Strategy for the Fractional-Order Time-Delay Multivariable Grey Prediction Model

2.1 Collaborative Identification Method for Fractional Order and Multivariable Time-Delay Lengths

Two important nonlinear parameters for the rationality of our model structure include: the fractional order, and the length of time-delay on each driving variable. Their combination affects both the estimation accuracy of other linear parameters as well as the prediction performance of the model in general. One decides the smoothness of the accumulated generation sequence while another one decides the temporal alignment of the causal relationship between the driving variables with the behavioral variable. They couple to some extent—different choices of orders changes the waveform morphology of the sequences which influences the result of the similarity measurement between sequences when identifying the time-delay. We thus need to use a collaborative identification strategy so they could be put together in a unified optimization framework simultaneously solved^[3].

The essence of collaborative identification is to develop an objective function which could fully reflect the model's fitting accuracy and prediction stability by using its value as a common metric to evaluate the relative merits between different parameter combinations. Because the fractional order typically has continuous values over the range of real numbers while the time-delay length usually varies from a set of discrete non-negative integers, there comes some degree of complexity when optimizing them with mixed continuous/discrete nature. A potential strategy may consider combining both grid search and numerical optimization strategies; scan through the parameter space on a relatively coarse grid to find candidates then refine the optimization algorithm locally inside those candidates.

The essence of collaborative identification is searching out (jointly on the parameter space between the order and the time delay), what makes the prediction performance of the model become optimal as changing of the order affects the smoothness foundation for time-delay identification; choosing the time delay will influence the structure of the model's driving terms in optimization of the order. We have to deal with them collaboratively to get a parameter configuration matching the intrinsic patterns of the data truly.

2.2 Least Squares Estimation of Model Structural Parameters and Its Improvement

Once the fractional order and time-delay length have been determined for all driving variables, other system development coefficient and driving coefficients corresponding to each driving variable in our model can also be linear parameters which were estimated using the least squares criterion. By converting the whitening equation of the model into discrete difference form, we get a set of linear equations about unknown linear parameters. The least squares estimation gets an optimal solution of the parameters which minimizes the sum of square fitting residuals, there exists a closed-form solution of it, thus give us relatively high computing efficiency on parameter estimations. However, traditional least squares estimation is very sensitive to anomalous observations in the data. When there exist outliers in the original sequences or when the noise distribution is uneven, the result of least squares estimation will be too pulled by anomalous values so that the parameter estimate deviate from its true value.

In order to solve this problem, we may use a weighting strategy (or regularization) to improve the least squares approach. The weighted least squares method assigns different weight to the data at different times based on its importance/reliability when fitting models; give small weights to those observations which have large residual so that they do not affect much. Regularization limits the norm of parameter vector with a penalty term for the parameters added into the objective function. When there are strong correlation between driving variables, regularizations will effectively relieve the instability in structural parameter estimation due to multicollinearity. Improving parameter estimation has advantages of simple computation like least squares, but increase the robustness of the model against non-ideal data condition. The quality of linear parameter estimation depends upon if our model could correctly capture the direction and strength of the influence of each driving variable on the behavioral variable. By adopting an appropriate improvement strategy, the parameter estimation result would be able to better reflect the true quantitative relationship among variables^[4].

2.3 Recursive Dynamic Parameter Adjustment Scheme for Multi-Step Prediction

An issue with applying our model in multi-step prediction is balancing between the fixity of

parameters versus the variability of the system over time; i.e., it might happen that the fixed parameters derived from historical period data do not represent the dynamics of the system during the prediction horizon if the number of steps of predictions increase. We believe that given the gradual gathering of new observations we have an opportunity for leveraging these new observations by adjusting the current parameters such that the prediction bias caused by parameter aging can be avoided. The core idea behind the proposed recursive dynamic parameter adjustment scheme is integrating the parameter update into the rolling prediction process whereupon availability of a new actual observation, the model's parameters will be updated ensuring them to always capture the latest operating state of the system.

How big should be the size of the update window and what percent of new/old data would we utilize in the updating process? Fixed sized window keeps only the last few period's data to re-estimate parameters by dropping off the older historical data which enables us to quickly track the structural changes in parameters but can lead to high estimation variance if not enough data exists. Fading memory uses all historical data emphasizing on the recent ones. We adopt a forgetting factor that controls the decay rate of the historical information so as to smoothly balance between stability and adaptability of parameters. The choice of updating mechanism depends upon our previous judgment about the speed of change of system parameters as well as actual need for estimating accuracy. With introduction of recursive updating scheme, the model has capability to absorb new information continuously during long-term prediction (single modeling or rolling modeling), achieving dynamic adaptation at parameter level without changing the basic structure of the model.

3. Performance Analysis and Applicable Boundary of the Fractional-Order Time-Delay Multivariable Grey Prediction Model

3.1 Mathematical Evaluation of Model Fitting Accuracy and Prediction Robustness

Model fitting accuracy reflects the ability to reproduce the historical data which is a basic index of the rationality of the model structure. In order to estimate the fitting accuracy, we have to take into account the aspects of absolute error and relative error. Absolute error could reflect the magnitude difference between the predicted value and the actual value whereas relative error eliminates the effect of dimensions so that comparing the accuracy among different sequences become commensurable. A single error indicator only represents an aspect of the model performance while multiple combination of error indicators will be able to better represent the model's fitting performance. Prediction robustness means the degree of consistency of the prediction results when the model faced some small fluctuation on data. Even though a model with inadequate robustness has good performance during the historical fitting process, after the original sequence slightly changed, there might be considerable deviation in its prediction results. We can evaluate the robustness by superimposing random disturbances of various intensities upon the original sequence and observe the sensitivity of the model output to those disturbances. Less sensitive to the model implies high capability to suppress external interference. Note: Fitting accuracy and prediction robustness are not always positively correlated. Sometimes, over-pursuing the fitting accuracy would sacrifice the robustness. When evaluating models, we need to pay attention to both indicators and find a reasonable balance between them^[5].

3.2 Decomposition of the Improvement Contributions of Introducing Fractional Order and Time Delay to the Traditional Grey Model

To have a better understanding about how much real impact on improving model performance brought by fractional-order accumulation and time-delay parameters, we need to construct some simplified versions of them for comparisons. We can build three different kinds of comparison models: 1st is a fractional-order multivariable model which doesn't contain any time delay (separating out the improvement effect of fractional-order accumulation); 2nd is a time-delay multivariable model with no fractional order (separating out the improvement effect of introducing time delay). And finally, an integer-order multivariable model without time delay as baseline. Doing such item-by-item stripping, we could decompose the performance gain of our model proposed in this paper into three independent parts -- the contribution of fractional-order accumulation; the contribution of introducing time delay and the contribution of their synergy. This kind of decomposition analysis would help us answering whether the improvement in model performance mostly came from optimizing the accumulation method, correcting the lag structure or adding extra gain due to the synergistic effect of both.

Theoretical viewpoint: The main embodiment of contribution of fractional-order accumulation is improving data smoothness (especially when the fluctuation amplitude of the original sequence is relatively large), while the main manifestation of introducing time-delay is correcting the temporal relationship among variables that dominate significantly if the transmission delay between driving variables and the behavioral variable are relatively obvious; interaction effect of both refers to whether the smoothing sequence facilitate identify time-delay accurate or whether the driving structure after time-delay correction make the optimization of the fractional order more reasonable. If it's positive, then we have a mutual reinforcement relationship with each other as a positive relationship in our model, but otherwise, there is some overlapping or competitive information extraction between the two mechanisms. We could use decomposition results to further simplify or extend the model structure, and will also help us understand how different improvement measures improve prediction performance.

3.3 Definition of Applicable Conditions of the Model under Different Data Characteristics

The fractional-order time-delay multivariable grey prediction model cannot be applied to all kinds of data sequences: it has a close relation with the inherent features of the data. Different sequences' lengths, rates of changes, fluctuation amplitudes, noises structure etc., affect the identification accuracy of the fractional order and time-delay parameters resulting in the influence on the final prediction performance of this model. If the sequence length is relatively short, there will be few degrees of freedoms for estimating parameters so as to make the probability of model overfitting increase; if the rate of change of the sequence is too fast, the smoothing effect of fractional-order accumulation might not be enough to remove the nonlinear parts from the trend; if the fluctuation amplitude of the sequence is too large or the noise level is relatively high, the similarity measure between sequences used for time-delay identification can be disturbed because of noise interference. Limitations caused by data characteristics on model performance is the fundamental dimension of the model's applicable boundary^[6].

We summarize here some conditions which make our model performs excellently and other conditions which make the model perform poorly for multiple aspects of data characteristics according to the above analysis: When the sequence length is enough and the trend of change is relatively smooth, it can fully play out the advantage of fractional-order accumulation in finely regulating smoothness, and the identification of time-delay parameters also have relatively high reliability; if there are obvious structural breakpoints in the sequence or the trend of change makes a sharp turn, then even the grey model with a fixed structure will face great challenges, introducing fractional order and time delay cannot fundamentally solve these problems. If the model's performance is limited, you could try pre-processing the original sequence to lower down the noise level or making targeted changes to the model structure. Defining the applicable conditions gives modelers an idea of choosing when they meet different data characteristics so as to avoid the misuse or abuse of the model.

Conclusion

The theory of the fractional-order time-delay multivariable grey prediction model was investigated on three aspects including model construction, parameter optimization and performance analysis. In the construction mechanism, the fractional-order accumulating generation flexibly adjusts the sequence smoothness by continually adjusting the order while the time-delay parameters can well reflect the causal time-delay structure within the multivariable driving information. Two kinds of mechanisms cooperated with the order-of-accumulation-first-and-time-delay-second way which leads to a clear structure of the model equation. In the parameter optimization, the collaborative identification method overcome the problem of synchronously solve the order and multivariable time delays in a joint space, the improved least squares estimation improves the robustness of linear parameters under non-ideal data condition, the recursive dynamic adjustment scheme give the model an adaptive parameter updating ability when making multi-step prediction. In the performance analysis, we establish a mathematical evaluation framework for the fitting accuracy and prediction robustness, clarify the origin of gains of the two types of mechanisms and their interaction via decomposing the contributions of improvements, define the applicable boundary of the model based on different aspects such as data length, rate of change, fluctuation amplitude, and noise level. We could generalize it into higher-order multivariable structures to describe more complicated system dynamics or explore time-varying settings of the fractional order so that the accumulation intensity adapts itself along the prediction process in our future works. Also, we intend to add nonlinear driving terms to improve the model's response capability against abrupt changes and turning points.

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